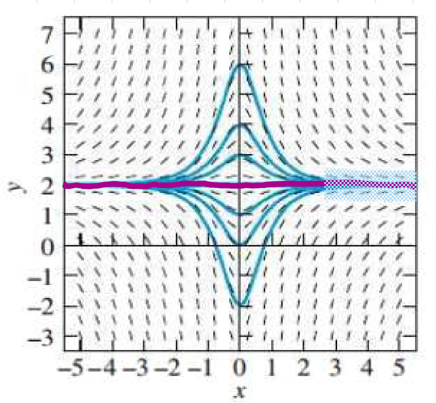
recall from last time:

find general solution of $(x^2+1)\frac{dy}{dx} + 3xy = 6x$



$y=2$

$y(x) = 2 + C(x^2+1)^{-3/2}$

as $x \to +\infty$ $\quad y \to 2$

when $C=0$: $\quad y(x) = 2$

also $y(0)=2$

$y(x) = 2$ for all x

called <u>equilibrium</u> solution (not dependent on x)

solution is horizontal line $m = y' = 0$

recall: closed intervals $[a, b]$

also: open intervals (a, b) endpoints NOT included

either could be $\pm\infty$

for a DE, an open interval I is a set of x-values s.t.:

- a solution exists
- solution is differentiable
- DE is defined

i.e., $y(x)$ and DE "make sense" for all x in I

<u>Theorem</u>

if $P(x)$ and $Q(x)$ are <u>continuous</u> functions on open interval, I, and I contains x_0 then initial value problem $\frac{dy}{dx} + P(x)y = Q(x)$ given $y(x_0) = y_0$ has a <u>unique</u> solution $y(x)$ on I

given by: $y(x) = e^{-\int P(x)dx}\left[\int \left(Q(x)e^{\int P(x)dx}\right)dx + C\right]$

using $y(x_0) = y_0$ and define $\rho(x) = e^{\int_{x_0}^{x} P(t)dt}$

using $y(x_0) = y_0$ and define $\rho(x) = e^{\int_{x_0}^{x} P(t)\,dt}$

can refine solution: $\frac{1}{\rho(x)}\left[\int_{x_0}^{x} Q(t)\,\rho(t)\,dt + y_0\right]$

ex. solve $x^2 \frac{dy}{dx} + xy = \sin x$ given $y(\underset{x_0}{1}) = y_0$

$$\frac{dy}{dx} + \underbrace{\frac{1}{x}}_{P(x)} y = \underbrace{\frac{\sin x}{x^2}}_{Q(x)}$$

integrating factor:

$$e^{\int_{x_0}^{x} P(t)\,dt}$$

$$= e^{\int_1^x \frac{1}{t}\,dt}$$

$$= e^{\ln|t|\,\Big|_1^x}$$

$$= e^{\ln x - \cancel{\ln 1}^{\,0}}$$

$$= x = \rho(x) \Rightarrow \rho(t) = t$$

$$y(x) = \frac{1}{x}\left[\int_1^x \frac{\sin t}{t^2}\, t\,dt + y_0\right]$$

$$\boxed{y(x) = \frac{1}{x}\left[\int_1^x \frac{\sin t}{t}\,dt + y_0\right]}$$ ← can stop here

Section 1.6 Substitution Methods

so far, have seen DE that were either separable or linear

ex. solve $\frac{dy}{dx} = (x+y+3)^2$

let $v = x+y+3 \Rightarrow y = v - x - 3$

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\frac{dv}{dx} - 1 = v^2$$

$$v^2 + 1 = \frac{dv}{dx}$$

$$dx = \frac{1}{1+v^2}\,dv \quad \text{integrate}$$

$$x = \arctan v + C$$

$$x - C = \arctan v$$

$$\tan(x-C) = v$$

$$\tan(x-C) = x+y+3$$

$\tan(x-C) = x+y+3$

$$\boxed{\tan(x-c) - x - 3 = y(x)}$$

ex. solve $2xy\frac{dy}{dx} = 4x^2 + 3y^2$

- not separable
- not linear

divide by $2xy$

instead, recognize as a homogenous equation :

has a format $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

$\leftarrow v$

let $v = \frac{y}{x} \Rightarrow y = vx$

$$\frac{dy}{dx} = \frac{dv}{dx}x + v$$

$$F(v) = x\frac{dv}{dx} + v \Rightarrow x\frac{dv}{dx} = F(v) - v \leftarrow \text{separable}$$

$$\frac{dy}{dx} = \frac{4x^2}{2xy} + \frac{3y^2}{2xy}$$

since $v = \frac{y}{x}$ also $\frac{1}{v} = \frac{x}{y}$

$$\frac{dy}{dx} = 2\frac{x}{y} + \frac{3}{2}\frac{y}{x}$$

$$\frac{dv}{dx}x + v = 2.\frac{1}{v} + \frac{3v}{2}$$

$$x\frac{dv}{dx} = \frac{2}{v} + \underbrace{\frac{3v}{2} - v}$$

$$x\frac{dv}{dx} = \frac{2}{2}\frac{2}{v} + \frac{1}{2}v.\frac{v}{v} = \frac{4+v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{4+v^2}{2v}$$ now separate

$$\int \frac{2v}{4+v^2}\,dv = \int \frac{1}{x}\,dx$$

$\ln|4+v^2| = \ln|x| + C$ where C is arb. constant

$\ln(4+v^2) = \ln|x| + C$ take e of both sides (exponentiate)

$4+v^2 = e^{\ln|x|}e^C$

let $A = e^C$

$v = \frac{y}{x} \downarrow \frac{1}{v} = \frac{x}{y}$

$$x^2\left(4 + \frac{y^2}{x^2} = A|x|\right)$$

$$4x^2 + y^2 = A\,|x^3|$$

let $k = A$ s.t. k is non-zero

$4x^2 + y^2 = k x^3$

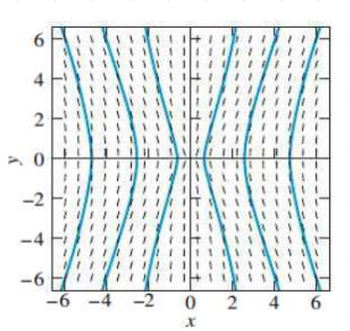
rewrite to expand eqn beyond

$$4x^2 + y^2 = k x^3$$

let $k = A$ s.t. k is non-zero
rewrite to expand eqn beyond bounds of ln

LHS ≥ 0
and $x \neq 0$ and $y \neq 0$ from definition of v (and subsequently $\frac{1}{v}$)

$\to$ LHS > 0 then if $x < 0$ $k < 0$
if $x > 0$ $k > 0$



solve for y: $y^2 = kx^3 - 4x^2$

$$y(x) = \pm\sqrt{kx^3 - 4x^2}$$

$$kx^3 - 4x^2 \geq 0$$
$$x^2(kx - 4) \geq 0$$
$x^2 > 0$ $\quad kx - 4 \geq 0$

$x > \frac{4}{k}$ $\quad$ $x < \frac{4}{k}$

$k > 0$ $\quad$ $k < 0$

ex. solve $x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}$ when $y(x_0) = 0$ where $x_0 > 0$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2 - y^2}}{x}$$

rework $\quad \frac{\sqrt{x^2 - y^2}}{\sqrt{x^2}} = \sqrt{\frac{x^2 - y^2}{x^2}}$

$v = \frac{y}{x}$

$y = vx$

$\frac{dy}{dx} = \frac{dv}{dx}x + v$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2}$$

$$\frac{dv}{dx}x + \cancel{v} = \cancel{v} + \sqrt{1 - v^2}$$

$$\frac{dv}{dx}x = \sqrt{1 - v^2}$$ separable :)

$$\int \frac{1}{\sqrt{1 - v^2}}\,dv = \int \frac{1}{x}\,dx$$

$$\arcsin v = \ln|x| + C$$ where C is an arb. constant

$$\arcsin \frac{y}{x} = \ln|x| + C$$

$$\frac{y}{x} = \sin(\ln|x| + C)$$

$$y(x) = x \sin(\ln|x| + C)$$

$$y(x) = x\sin(\ln|x| + C)$$

when $y(x_0) = 0$

given $x_0 > 0$

$$0 = x_0 \sin(\ln x_0 + C)$$

$x_0 = 0$
invalid sol'n

$$\sin(\ln(x_0) + C) = 0$$

$$\ln(x_0) + C = \cancel{\arcsin 0} \; 0$$

$$\ln(x_0) + C = 0$$

$$C = -\ln(x_0)$$

$$y(x) = x\sin(\ln|x| - \ln(x_0))$$

assume $x > 0$
near $x = x_0 > 0$

$$\boxed{y(x) = x\sin\left(\ln\frac{x}{x_0}\right)}$$